Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Journal of Sound and Vibration 274 (2004) 915–938
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Stochastic vibration of axially loaded monosymmetric Timoshenko thin-walled beam

Jun Li*, Hongxing Hua, Rongying Shen, Xianding Jin

Vibration, Shock and Noise Institute, Shanghai Jiao Tong University, 200030 Shanghai, People's Republic of China

Received 28 January 2003; accepted 2 June 2003

Abstract

An analytical method is presented to perform the flexure–torsion coupled stochastic response analysis of monosymmetric axially loaded Timoshenko thin-walled beam subjected to various kinds of concentrated and distributed stochastic excitations with stationary and ergodic properties. The effects of warping stiffness, axial force, shear deformation and rotary inertia are included in the present formulations. First, the damped general governing differential equations of motion of axially loaded Timoshenko thin-walled beam are developed and the free vibration analysis is performed. Once the natural frequencies and mode shapes are obtained, mode superposition method in conjunction with receptance method is used to compute the mean square displacement response of the axially loaded thin-walled beam. Finally, the method is illustrated by its application to two test examples to investigate the effects of warping stiffness, axial force, shear deformation and rotary inertia on the stochastic response of the thin-walled beams.

1. Introduction

The thin-walled beam structures are commonly found in the design of the aircraft wings, propeller blades, bridge decks, vehicles axles and so on, due to their outstanding properties. Since the thin-walled beam members are widely used in aerospace, automobile and civil architecture industries, it is important to ensure that their design is reliable and safe. The dynamic analyses of the thin-walled beam structures also help to optimize the design and avoid future investments on repairs. It is, therefore, essential for design engineers to evaluate the dynamic characteristics of the thin-walled beam structures accurately.

*Corresponding author.
E-mail address: ljy60023@sohu.com (J. Li).

0022-460X/$ - see front matter
doi:10.1016/S0022-460X(03)00654-0

It is well known that when the cross-section of the beam has two symmetric axes, the shear center and the centroid of the cross-section coincide, and all flexural and torsional vibrations are independent of each other, this case represents no coupling at all. Then the classical Bernoulli–Euler and/or the Timoshenko beam theory are valid. However, for a large number of practical beams of thin-walled sections, the centroid and shear center of the cross-section are obviously non-coincident, the above assumption is not valid. When the cross-section of the thin-walled beam has only one symmetrical axis, the flexural vibration in the direction of the symmetrical axis is independent of the other vibrations. But the flexural vibration in the perpendicular direction of the symmetric axis is coupled with torsional vibration.

In spite of the practical interest of the flexure–torsion coupled thin-walled beam problems, particularly in the context of aerospace, civil and mechanical applications, the main body of the available investigations has devoted entirely to study the dynamic response of beams having double symmetrical axes and structures composed of this kind of beams such as Refs. [1–4]. There is a number of research works dealing with the free vibration characteristics of the flexure–torsion coupled thin-walled beams [5–20]. Bishop et al. [5] studied the coupled flexural–torsional vibration of the Timoshenko beam without the warping stiffness included. Hallauer et al. [6] and Friberg [7] derived the exact dynamic stiffness matrix for a flexure–torsion coupled Bernoulli–Euler beam with the warping stiffness ignored. Bishop et al. [10] extended the work of Dokumaci [9] to include the warping stiffness term. Friberg [8] and Leung [12,13] developed the dynamic stiffness matrix of a Vlasov beam with the shear deformation completely ignored. Dvorkin et al. [11] presented a traditional finite element formulation of the similar problem. Banerjee et al. [14,15] derived the analytical expressions for the coupled flexural–torsional dynamic stiffness matrix of an axially loaded Timoshenko beam excluding the warping stiffness effect. Klausbruckner et al. [16] investigated the vibration characteristics of the channel beams based on theoretical and experimental study. Banerjee et al. [17] formulated an exact dynamic stiffness matrix for a Bernoulli–Euler thin-walled beam with inclusion of the warping stiffness. Bercin et al. [18] presented the coupled flexural–torsional vibration of the Timoshenko beam with warping stiffness included. Tanaka et al. [19] presented the exact solution for the flexure–torsion coupled Bernoulli–Euler beam including the warping stiffness. Hashemi et al. [20] presented a new dynamic finite element for the axially loaded flexure–torsion coupled Bernoulli–Euler beam with the warping stiffness omitted. All of the above studies only investigated the eigenvalue problems of the thin-walled beams. Relatively fewer studies [21,22] are available that have been done toward the study of the forced vibration response of the thin-walled beams subjected to deterministic or stochastic external excitations. Chen et al. [21] employed the finite element method in conjunction with an implicit-starting unconditionally stable methodology for the dynamic computation of elastic open section thin-walled structures subjected to deterministic loads. The paper employed Vlasov's assumptions and both warping and rotary inertia are included in the developments. But two important design parameters, namely the effects of axial force and shear deformation, were not included in the formulations and the paper focused attention on the deterministic dynamic response in the time domain. Eslimy-Isfahany et al. [22] developed an analytical theory to investigate the response of a flexure–torsion coupled beam to deterministic and stochastic excitations by using the normal mode method. Unfortunately, the authors assumed that the beam twists according to the Saint–Venant theory and thus no allowance is made for warping stiffness of the beam cross-section. Such an assumption can lead to large errors when calculating the dynamic response of a thin-walled open section beam. Also the effects of axial force, shear deformation and rotary inertia were not included in the formulation.

In order to accurately predict the stochastic response of the thin-walled beam structures, comprehensive structural models have to be used. In particular, shear deformation, rotatory inertia, warping stiffness as well as flexure–torsion coupling and axial force must be included in their modelling. The necessity of incorporating transverse shear effect arises from the fact that it is usually important when the depth-span ratio of the thin-walled beam is relatively large. In addition, torsion related non-uniform warping occurs when a section is restrained against out of plane deformation and/or when a non-uniform distributed torque is applied along the length of the beam. Therefore, the free warping assumption may result in erroneous predictions of the behavior of cantilevered type structures. Consequently, the warping stiffness effect is incorporated in this work.

In the previous studies of the stochastic response analysis of the flexure–torsion coupled thin-walled beams, it seems that there is no work investigating the effects of axial force, warping stiffness, shear deformation and rotatory inertia on the dynamic behavior of the thin-walled beams simultaneously. This problem is presented in this paper. The stochastic flexure–torsion coupled vibration of elastic axially loaded thin-walled beams with monosymmetrical cross-section subjected to various kinds of concentrated and distributed stochastic excitations with stationary and ergodic properties is investigated. The effects due to axial force and warping stiffness on the stochastic response of the thin-walled beams are especially interested. Furthermore, the effects due to shear deformation and rotatory inertia are also of interest here. First, an analytical method for determining natural frequencies and mode shapes of the flexure–torsion coupled vibration of the axially loaded beams with thin-walled monosymmetrical cross-section is developed by using the general solution of the governing differential equations of motion. This method takes into account the effects of axial force, warping stiffness, shear deformation and rotatory inertia in a unified way. Once the natural frequencies and mode shapes are obtained, normal mode method in conjunction with receptance method is used to compute the mean square displacement response of the axially loaded thin-walled beam. Finally, the method is illustrated by its application to two test examples to investigate the effects of warping stiffness, axial force, shear deformation and rotary inertia on the stochastic response of the thin-walled beams.

2. Free vibration of axially loaded Timoshenko thin-walled beam including warping effect

Under consideration is a thin-walled structure, modelled as an open or closed thin-walled beam with monosymmetrical cross-section. Considering a uniform and straight thin-walled beam with length L shown in Fig. 1, it is assumed that the terms associated with secondary warping and warping inertia which are negligibly small can be discarded. The shear center and centriod of the thin-walled beam are denoted by s and c, respectively, which are separated by distance y_c. In the right handed Cartesian co-ordinate system in Fig. 1, the x-axis is assumed to coincide with the elastic axis (i.e., loci of the shear center of the cross-section of the thin-walled beam). The flexural translation in the z direction and the torsional rotation about the x-axis of the shear center are denoted by $v(x, t)$ and $\psi(x, t)$, respectively, where x and t denote distance from the origin and time, respectively. The rotation of the elastic axis due to flexure alone is denoted by $\theta(x, t)$. A constant

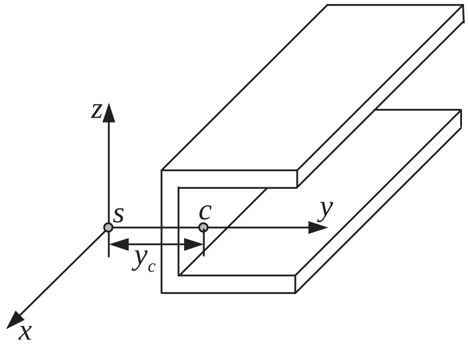

Fig. 1. An axially loaded Timoshenko thin-walled beam.

compression axial force P is assumed to act through the centroid of the cross-section of the thin-walled beam. P can be positive or negative, so that tension is included. The external excitations acting on the thin-walled beam are represented by a force $f(x,t)$ per unit length that parallel to sz-axis and applied to the shear center together with a torque $m(x,t)$ per unit length about sx-axis, respectively.

The damped governing equations of motion for the forced vibration of the axially loaded Timoshenko thin-walled beam exhibiting flexure–torsion coupling and including warping stiffness effect can be written as following three coupled differential equations, which may be derived using Hamilton's principle (for details of the derivation, see Appendix A)

$$\rho I\ddot{\theta} + c_3\dot{\theta} - EI\theta'' - kAG(v' - \theta) = 0, \tag{1}$$

$$I_s\ddot{\psi} - \mu y_c\ddot{v} + c_2\dot{\psi} - c_1 y_c\dot{v} - GJ\psi'' + P(I_s\psi''/\mu - y_c v'') + E\Gamma\psi'''' = m(x,t), \tag{2}$$

$$\mu\ddot{v} + c_1(\dot{v} - y_c\dot{\psi}) - \mu y_c\ddot{\psi} - kAG(v'' - \theta') + P(v'' - y_c\psi'') = f(x,t), \tag{3}$$

where E is Young's modulus of elasticity of the thin-walled beam material, G is the shear modulus of the thin-walled beam material. EI, kGA, GJ and $E\Gamma$ are flexural stiffness, shear stiffness, torsional stiffness and warping stiffness of the thin-walled beam, respectively. μ is mass of the thin-walled beam per unit length, I is the second area moment of inertia of the beam cross-section about y-axis, I_s is polar mass moment of inertia of per unit length thin-walled beam about x-axis, superscript primes and dots denote the derivative with respect to position x and time t, respectively. ρ is the density of the thin-walled beam material, A is the cross-section area of the thin-walled beam, k is the effective area coefficient in shear. The damping coefficients c_1, c_2 and c_3 are the linear viscous damping terms of per unit length thin-walled beam in flexural deformation, torsional deformation and rotatory deformation due to flexure, respectively.

For undamped free vibration of axially loaded Timoshenko thin-walled beam, the external excitations $f(x,t)$ and $m(x,t)$ are set to zero, as are the damping coefficients c_1, c_2 and c_3, in order to determine the natural frequencies and mode shapes of the thin-walled beam. A sinusoidal variation of $v(x,t)$, $\theta(x,t)$ and $\psi(x,t)$ with circular frequency ω_n is assumed to be of the forms

$$v(x,t) = V_n(x)\sin\omega_n t, \tag{4}$$

$$\theta(x,t) = \Theta_n(x)\sin\omega_n t, \tag{5}$$

$$\psi(x,t) = \Psi_n(x)\sin\omega_n t, \tag{6}$$

where $n = 1, 2, 3, \ldots$, $V_n(x)$, $\Theta_n(x)$ and $\Psi_n(x)$ are the amplitudes of the sinusoidally varying flexural translation, flexural rotation and torsional rotation, respectively.

Substituting Eqs. (4)–(6) into Eqs. (1)–(3) gives the three simultaneous differential equations for V_n, Θ_n and Ψ_n.

$$\rho I\omega_n^2\Theta_n + EI\Theta_n'' + kGA(V_n' - \Theta_n) = 0, \tag{7}$$

$$GJ\Psi_n'' - P(I_s\Psi_n''/\mu - y_cV_n'') + I_s\omega_n^2\Psi_n - \omega_n^2\mu y_cV_n - E\Gamma\Psi_n'''' = 0, \tag{8}$$

$$kGA(\Theta_n' - V_n'') + P(V_n'' - y_c\Psi_n'') - \mu\omega_n^2V_n + \mu y_c\omega_n^2\Psi_n = 0. \tag{9}$$

After extensive algebra manipulation, Eqs. (7)–(9) can be combined into one equation by either eliminating all but one of the three variables V_n, Θ_n and Ψ_n to give the following eighth order differential equation:

$$\begin{aligned}&\{(d-psd)D^8 + (-p^2a_ncs/b_n + pa_n/b_n + ps + pd - b_ndrps + b_nd(s+r) - 1)D^6,\\ &+ (a_nrp - p + b_nrps + p^2a_nc/b_n - a_ncp^2rs + 2a_ncps - b_n(s+r+d-b_nsrd) - a_n)D^4,\\ &+ (-2a_ncp + 2a_nb_ncrps - b_n(a_nr + b_nsr - 1 + a_ncs))D^2 + a_ncb_n(1-b_nrs)\}\mathrm{X}_n = 0,\end{aligned} \tag{10}$$

where

$$\mathrm{X}_n = V_n,\ \Theta_n \text{ or } \Psi_n,\quad D = \mathrm{d}/\mathrm{d}\xi,\quad \xi = x/L,\quad a_n = I_s\omega_n^2L^2/GJ,\quad b_n = \mu\omega_n^2L^4/EI,$$
$$c = 1 - \mu y_c^2/I_s,\quad d = E\Gamma/GJL^2,\quad r = I/AL^2,\quad s = EI/kAGL^2,\quad p = PL^2/EI.$$

Note that r, s and p above describe the effects of rotatory inertia, shear deformation and axial force, respectively. Any one or all of these parameters can be set to zero so that corresponding effect(s) can be optionally ignored.

The solution of the differential equation (10) can be obtained by substituting the trial solution $X_n = \mathrm{e}^{\kappa_n\xi}$ to give the characteristic equation

$$\begin{aligned}&(d-psd)\kappa_n^8 + (-p^2a_ncs/b_n + pa_n/b_n + ps + pd - b_ndrps + b_nd(s+r) - 1)\kappa_n^6\\ &\quad + (a_nrp - p + b_nrps + p^2a_nc/b_n - a_ncp^2rs + 2a_ncps - b_n(s+r+d-b_nsrd) - a_n)\kappa_n^4\\ &\quad + (-2a_ncp + 2a_nb_ncrps - b_n(a_nr + b_nsr - 1 + a_ncs))\kappa_n^2 + a_ncb_n(1-b_nrs) = 0.\end{aligned} \tag{11}$$

Let

$$\chi_n = \kappa_n^2. \tag{12}$$

Substituting Eq. (12) into Eq. (11) gives

$$\begin{aligned}&(d-psd)\chi_n^4 + (-p^2a_ncs/b_n + pa_n/b_n + ps + pd - b_ndrps + b_nd(s+r) - 1)\chi_n^3\\ &\quad + (a_nrp - p + b_nrps + p^2a_nc/b_n - a_ncp^2rs + 2a_ncps - b_n(s+r+d-b_nsrd) - a_n)\chi_n^2\\ &\quad + (-2a_ncp + 2a_nb_ncrps - b_n(a_nr + b_nsr - 1 + a_ncs))\chi_n + a_ncb_n(1-b_nrs) = 0.\end{aligned} \tag{13}$$

It can be shown [10] that all four roots of Eq. (13) are real, two of them negative and the other two positive. Suppose that the four roots are χ_{n1}, χ_{n2}, $-\chi_{n3}$, $-\chi_{n4}$, where χ_{nj} $(j = 1, \ldots, 4)$ are real and positive. Then the eight roots of the characteristic Eq. (11) are

$$\alpha_n, -\alpha_n, \beta_n, -\beta_n, \mathrm{i}\gamma_n, -\mathrm{i}\gamma_n, \mathrm{i}\delta_n, -\mathrm{i}\delta_n.$$

where $\mathrm{i} = \sqrt{-1}$ and $\alpha_n = \sqrt{\chi_{n1}}$, $\beta_n = \sqrt{\chi_{n2}}$, $\gamma_n = \sqrt{\chi_{n3}}$, $\delta_n = \sqrt{\chi_{n4}}$.

It follows that the solution of the Eq. (10) is of the following form:

$$\begin{aligned} V_n(\xi) = {} & c_1^* \cosh \alpha_n\xi + c_2^* \sinh \alpha_n\xi + c_3^* \cosh \beta_n\xi + c_4^* \sinh \beta_n\xi \\ & + c_5^* \cos \gamma_n\xi + c_6^* \sin \gamma_n\xi + c_7^* \cos \delta_n\xi + c_8^* \sin \delta_n\xi, \end{aligned} \tag{14}$$

$$\begin{aligned} \Psi_n(\xi) = {} & t_{n1}c_1^* \cosh \alpha_n\xi + t_{n1}c_2^* \sinh \alpha_n\xi + t_{n2}c_3^* \cosh \beta_n\xi + t_{n2}c_4^* \sinh \beta_n\xi \\ & + t_{n3}c_5^* \cos \gamma_n\xi + t_{n3}c_6^* \sin \gamma_n\xi + t_{n4}c_7^* \cos \delta_n\xi + t_{n4}c_8^* \sin \delta_n\xi \end{aligned} \tag{15}$$

$$\begin{aligned} \Theta_n(\xi) = {} & t_{n5}c_2^* \cosh \alpha_n\xi + t_{n5}c_1^* \sinh \alpha_n\xi + t_{n6}c_4^* \cosh \beta_n\xi + t_{n6}c_3^* \sinh \beta_n\xi \\ & + t_{n7}c_6^* \cos \gamma_n\xi - t_{n7}c_5^* \sin \gamma_n\xi + t_{n8}c_8^* \cos \delta_n\xi - t_{n8}c_7^* \sin \delta_n\xi, \end{aligned} \tag{16}$$

where c_1^*–c_8^* is a set of constants which can be determined from the boundary condition, and

$$t_{n1} = a_n(1-c)(b_n - p\alpha_n^2)/(a_nb_n - a_np\alpha_n^2 + b_n\alpha_n^2)y_c,$$

$$t_{n2} = a_n(1-c)(b_n - p\beta_n^2)/(a_nb_n - a_np\beta_n^2 + b_n\beta_n^2)y_c,$$

$$t_{n3} = a_n(1-c)(b_n + p\gamma_n^2)/(a_nb_n + a_np\gamma_n^2 - b_n\gamma_n^2)y_c,$$

$$t_{n4} = a_n(1-c)(b_n + p\delta_n^2)/(a_nb_n + a_np\delta_n^2 - b_n\delta_n^2)y_c,$$

$$t_{n5} = \alpha_n/L(1 - b_nrs - \alpha_n^2 s), \quad t_{n6} = \beta_n/L(1 - b_nrs - \beta_n^2 s),$$

$$t_{n7} = \gamma_n/L(1 - b_nrs + \gamma_n^2 s), \quad t_{n8} = \delta_n/L(1 - b_nrs + \delta_n^2 s).$$

Eqs. (14)–(16) in conjunction with the boundary conditions yield the eigenvalues (natural frequencies) and eigenfunctions (mode shapes) of axially loaded Timoshenko thin-walled beam. Also based on Eqs. (7)–(9) and the boundary conditions, by following the procedure described in Ref. [10], the following orthogonality for different mode shapes of the Timoshenko thin-walled beam can be derived as

$$\int_0^1 \{(\rho I\Theta_m\Theta_n + \mu V_mV_n + I_s\Psi_m\Psi_n) - \mu y_c(V_m\Psi_n + V_n\Psi_m)\}\, \mathrm{d}\xi = \bar{m}_n\delta_{mn}, \tag{17}$$

where $\bar{m}_n$ is the generalized mass in the nth mode, δ_{mn} is the Kronecker delta function.

With the free vibration modes, natural frequencies and orthogonality condition described above, it is now possible to investigate the general stochastic vibration problem of the damped axially loaded Timoshenko thin-walled beam.

3. Stochastic vibration analysis of axially loaded Timoshenko thin-walled beam including warping

For forced stochastic vibration of the axially loaded Timoshenko thin-walled beam, assume $v(x,t)$, $\theta(x,t)$, $\psi(x,t)$ can be expanded in terms of the eigenfunctions to give the following three equations:

$$v(x,t) = v(\xi L, t) = \sum_{n=1}^{\infty} q_n(t) V_n(\xi), \tag{18}$$

$$\psi(x,t) = \psi(\xi L, t) = \sum_{n=1}^{\infty} q_n(t) \Psi_n(\xi), \tag{19}$$

$$\theta(x,t) = \theta(\xi L, t) = \sum_{n=1}^{\infty} q_n(t) \Theta_n(\xi), \tag{20}$$

where $q_n(t)$ is the generalized time-dependent co-ordinate for each mode. Substituting Eqs. (18)–(20) into Eqs. (1)–(3) and using Eqs. (7)–(9) yields

$$\sum_{n=1}^{\infty}[\mu(V_n - y_c\Psi_n)\ddot{q}_n + c_1(V_n - y_c\Psi_n)\dot{q}_n + \mu\omega_n^2(V_n - y_c\Psi_n)q_n] = f(\xi,t), \tag{21}$$

$$\sum_{n=1}^{\infty}[\rho I\Theta_n\ddot{q}_n + c_3\Theta_n\dot{q}_n + \rho I\omega_n^2\Theta_n q_n] = 0, \tag{22}$$

$$\sum_{n=1}^{\infty}[(I_s\Psi_n - \mu y_c V_n)\ddot{q}_n + (c_2\Psi_n - c_1 V_n y_c)\dot{q}_n + \omega_n^2(I_s\Psi_n - \mu y_c V_n)q_n] = m(\xi,t), \tag{23}$$

where superscript dot denotes derivative with respect to time.

Multiplying Eqs. (21)–(23) by V_m, Θ_m and Ψ_m, respectively, then summing up these three equations and integrating from 0 to 1, and using orthogonality condition (17) gives

$$\ddot{q}_n(t) + 2\zeta_n\omega_n\dot{q}_n(t) + \omega_n^2 q_n(t) = [F_n(t) + M_n(t)], \tag{24}$$

where $F_n(t)$ and $M_n(t)$ can be expressed as

$$F_n(t) = \frac{1}{\bar{m}_n}\int_0^1 V_n(\xi)f(\xi,t)\,\mathrm{d}\xi, \quad M_n(t) = \frac{1}{\bar{m}_n}\int_0^1 \Psi_n(\xi)m(\xi,t)\,\mathrm{d}\xi,$$

$$\zeta_n = \frac{c_1}{2\mu\omega_n} = \frac{c_2}{2I_s\omega_n} = \frac{c_3}{2\rho I\omega_n},$$

where ζ_n is a non-dimensional quantity known as the viscous damping factor. Here the assumption $c_2 = c_1 I_s/\mu$, $c_3 = c_1\rho I/\mu$ has been made to take advantage of the orthogonality condition (17) in order to avoid having coupling terms $\dot{q}_n$ in Eq. (24).

In this paper, the stochastic response of the axially loaded Timoshenko thin-walled beam to stationary, ergodic stochastic excitations with zero initial conditions is investigated in the frequency domain by using the receptance method.

From Eq. (24), the cross-spectral density function $S_{q_nq_l}(\Omega)$ of the generalized time-dependent co-ordinate $q_n(t)$ is related to the cross-spectral density functions $S_{F_nF_l}(\Omega)$ and $S_{M_nM_l}(\Omega)$ of the flexure load $F_n(t)$ and torsional load $M_n(t)$ by the following relation:

$$S_{q_nq_l}(\Omega) = H_n^*(\Omega)[S_{F_nF_l}(\Omega) + S_{M_nM_l}(\Omega)]H_l(\Omega), \tag{25}$$

where $H_l(\Omega)$ is the receptance

$$H_l(\Omega) = \frac{1}{(\omega_l^2 - \Omega^2 + 2\mathrm{i}\zeta_l\Omega\omega_l)},$$

$H_n^*(\Omega)$ is the complex conjugate of $H_n(\Omega)$, $S_{F_nF_l}(\Omega)$ is the cross-spectral density function between $F_n(t)$ and $F_l(t)$, $S_{M_nM_l}(\Omega)$ is the cross-spectral density function between $M_n(t)$ and $M_l(t)$. Since it is assumed that the stochastic excitations $f(\xi, t)$ and $m(\xi, t)$ are stationary in time, then so are the generalized forces $F_n(t)$ and$M_n(t)$. Furthermore, $F_n(t)$and $M_n(t)$ are assumed to be independent stochastic processes so that the cross-spectral density functions between $F_n(t)$ and $M_n(t)$ can be excluded.

Based on the expressions of the generalized forces $F_n(t)$ and$M_n(t)$, the cross-spectral density functions $S_{F_nF_l}(\Omega)$ and $S_{M_nM_l}(\Omega)$ can be obtained explicitly as, respectively

$$S_{F_nF_l}(\Omega) = \frac{1}{\bar{m}_n\bar{m}_l}\int_0^1\int_0^1 V_n(\xi_1)V_l(\xi_2)S_f(\xi_1,\xi_2,\Omega)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2,$$

$$S_{M_nM_l}(\Omega) = \frac{1}{\bar{m}_n\bar{m}_l}\int_0^1\int_0^1 \Psi_n(\xi_1)\Psi_l(\xi_2)S_m(\xi_1,\xi_2,\Omega)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2, \tag{26}$$

where $S_f(\xi_1, \xi_2, \Omega)$ is the distributed cross-spectral density function between the stochastic flexural loads $f(\xi_1, t)$ and $f(\xi_2, t)$, $S_m(\xi_1, \xi_2, \Omega)$ is the distributed cross-spectral density function between the stochastic torsional loads $m(\xi_1, t)$ and $m(\xi_2, t)$.

For the flexural load $f(\xi, t)$ and torsional load $m(\xi, t)$, the corresponding cross-spectral density functions $S_f(\xi_1, \xi_2, \Omega)$ and $S_m(\xi_1, \xi_2, \Omega)$ are related to the cross-correlation functions $R_f(\xi_1, \xi_2, \tau)$ and $R_m(\xi_1, \xi_2, \tau)$, respectively, by the following Fourier transform pairs:

$$S_f(\xi_1,\xi_2,\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_f(\xi_1,\xi_2,\tau)\mathrm{e}^{-\mathrm{i}\Omega\tau}\,\mathrm{d}\tau, \quad R_f(\xi_1,\xi_2,\tau) = \int_{-\infty}^{\infty} S_f(\xi_1,\xi_2,\Omega)\mathrm{e}^{\mathrm{i}\Omega\tau}\,\mathrm{d}\Omega,$$

$$S_m(\xi_1,\xi_2,\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_m(\xi_1,\xi_2,\tau)\,\mathrm{e}^{-\mathrm{i}\Omega\tau}\,\mathrm{d}\tau, \quad R_m(\xi_1,\xi_2,\tau) = \int_{-\infty}^{\infty} S_m(\xi_1,\xi_2,\Omega)\,\mathrm{e}^{\mathrm{i}\Omega\tau}\,\mathrm{d}\Omega. \tag{27}$$

The cross-correlation functions $R_f(\xi_1, \xi_2, \tau)$ and $R_m(\xi_1, \xi_2, \tau)$ of the flexural load $f(\xi, t)$ and torsional load $m(\xi, t)$ are defined as

$$R_f(\xi_1,\xi_2,\tau) = E[f(\xi_1,t)f(\xi_2,t+\tau)], \quad R_m(\xi_1,\xi_2,\tau) = E[m(\xi_1,t)m(\xi_2,t+\tau)], \tag{28}$$

where $E[\,]$ denotes the ensemble average of the stochastic process.

According to Eqs. (18)–(20), the cross-spectral density functions $S_v(\xi_1, \xi_2, \Omega)$, $S_\psi(\xi_1, \xi_2, \Omega)$ and $S_\theta(\xi_1, \xi_2, \Omega)$ of the flexural translation $v(\xi, t)$, torsional rotation $\psi(\xi, t)$ and flexural rotation $\theta(\xi, t)$ can be written as

$$S_v(\xi_1,\xi_2,\Omega) = \sum_{n=1}^{\infty}\sum_{l=1}^{\infty} V_n(\xi_1)V_l(\xi_2)S_{q_nq_l}, \tag{29}$$

$$S_\psi(\xi_1, \xi_2, \Omega) = \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \Psi_n(\xi_1)\Psi_l(\xi_2)S_{q_n q_l}, \tag{30}$$

$$S_\theta(\xi_1, \xi_2, \Omega) = \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \Theta_n(\xi_1)\Theta_l(\xi_2)S_{q_n q_l}. \tag{31}$$

Substitution of Eq. (25) into Eqs. (29)–(31) gives

$$S_v(\xi_1, \xi_2, \Omega) = \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} V_n(\xi_1)V_l(\xi_2)H_n^*(\Omega)H_l(\Omega)[S_{F_n F_l}(\Omega) + S_{M_n M_l}(\Omega)], \tag{32}$$

$$S_\psi(\xi_1, \xi_2, \Omega) = \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \Psi_n(\xi_1)\Psi_l(\xi_2)H_n^*(\Omega)H_l(\Omega)[S_{F_n F_l}(\Omega) + S_{M_n M_l}(\Omega)], \tag{33}$$

$$S_\theta(\xi_1, \xi_2, \Omega) = \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \Theta_n(\xi_1)\Theta_l(\xi_2)H_n^*(\Omega)H_l(\Omega)[S_{F_n F_l}(\Omega) + S_{M_n M_l}(\Omega)]. \tag{34}$$

Now, substituting Eq. (26) into Eqs. (32)–(34), one obtains

$$S_v(\xi_1, \xi_2, \Omega) = \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} h_n^*(\Omega)h_l(\Omega)\eta_{nl}(\Omega)V_n(\xi_1)V_l(\xi_2), \tag{35}$$

$$S_\psi(\xi_1, \xi_2, \Omega) = \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} h_n^*(\Omega)h_l(\Omega)\eta_{nl}(\Omega)\Psi_n(\xi_1)\Psi_l(\xi_2), \tag{36}$$

$$S_\theta(\xi_1, \xi_2, \Omega) = \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} h_n^*(\Omega)h_l(\Omega)\eta_{nl}(\Omega)\Theta_n(\xi_1)\Theta_l(\xi_2), \tag{37}$$

where $h_l(\Omega) = 1/[\bar{m}_l(\omega_l^2 - \Omega^2 + 2\mathrm{i}\zeta_l\omega_l\Omega)]$:

$$\eta_{nl}(\Omega) = \int_0^1 \int_0^1 \{V_n(\xi_1)V_l(\xi_2)S_f(\xi_1, \xi_2, \Omega) + \Psi_n(\xi_1)\Psi_l(\xi_2)S_m(\xi_1, \xi_2, \Omega)\}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2.$$

For $\xi_1 = \xi_2 = \xi$, the cross-spectral density functions $S_v(\xi_1, \xi_2, \Omega)$, $S_\psi(\xi_1, \xi_2, \Omega)$ and $S_\theta(\xi_1, \xi_2, \Omega)$ reduce to the spectral density functions $S_v(\xi, \Omega)$, $S_\psi(\xi, \Omega)$ and $S_\theta(\xi, \Omega)$:

$$S_v(\xi, \Omega) = \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} h_n^*(\Omega)h_l(\Omega)\eta_{nl}(\Omega)V_n(\xi)V_l(\xi), \tag{38}$$

$$S_\psi(\xi, \Omega) = \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} h_n^*(\Omega)h_l(\Omega)\eta_{nl}(\Omega)\Psi_n(\xi)\Psi_l(\xi), \tag{39}$$

$$S_\theta(\xi, \Omega) = \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} h_n^*(\Omega)h_l(\Omega)\eta_{nl}(\Omega)\Theta_n(\xi)\Theta_l(\xi). \tag{40}$$

The mean square values of the flexural translation, torsional rotation and flexural rotation can be found by integrating the corresponding spectral density functions over all frequencies

$$E[v^2(\xi, t)] = \int_{-\infty}^{\infty} S_v(\xi, \Omega)\,\mathrm{d}\Omega, \tag{41}$$

$$E[\psi^2(\xi, t)] = \int_{-\infty}^{\infty} S_\psi(\xi, \Omega)\,\mathrm{d}\Omega, \tag{42}$$

$$E[\theta^2(\xi, t)] = \int_{-\infty}^{\infty} S_\theta(\xi, \Omega)\,\mathrm{d}\Omega. \tag{43}$$

If the external stochastic excitations are assumed to follow the Gaussian probability distribution, the response probability will also be Gaussian, and therefore the response can be fully described by its spectral density function.

As an example, two kinds of loads are considered here as applied on the thin-walled beam. The first one is that there is only one stochastic varying concentrated flexural load acting on the thin-walled beam at $\xi = \xi_f$. In this case, $\eta_{nl}(\Omega)$ in Eqs. (35)–(37) can be simplified as

$$\eta_{nl}(\Omega) = V_n(\xi_f)V_l(\xi_f)S_f(\Omega). \tag{44}$$

The spectral density functions of the flexural translation, torsional rotation and flexural rotation are then given by Eqs. (38)–(40) as

$$\begin{aligned} S_v(\xi, \Omega) &= \sum_{n=1}^{\infty}\sum_{l=1}^{\infty} h_n^*(\Omega)V_n(\xi)V_n(\xi_f)h_l(\Omega)V_l(\xi)V_l(\xi_f)S_f(\Omega) \\ &= \left|\sum_{l=1}^{\infty} h_l(\Omega)V_l(\xi)V_l(\xi_f)\right|^2 S_f(\Omega), \end{aligned} \tag{45}$$

$$\begin{aligned} S_\psi(\xi, \Omega) &= \sum_{n=1}^{\infty}\sum_{l=1}^{\infty} h_n^*(\Omega)\Psi_n(\xi)V_n(\xi_f)h_l(\Omega)\Psi_l(\xi)V_l(\xi_f)S_f(\Omega) \\ &= \left|\sum_{l=1}^{\infty} h_l(\Omega)\Psi_l(\xi)V_l(\xi_f)\right|^2 S_f(\Omega), \end{aligned} \tag{46}$$

$$\begin{aligned} S_\theta(\xi, \Omega) &= \sum_{n=1}^{\infty}\sum_{l=1}^{\infty} h_n^*(\Omega)\Theta_n(\xi)V_n(\xi_f)h_l(\Omega)\Theta_l(\xi)V_l(\xi_f)S_f(\Omega) \\ &= \left|\sum_{l=1}^{\infty} h_l(\Omega)\Theta_l(\xi)V_l(\xi_f)\right|^2 S_f(\Omega). \end{aligned} \tag{47}$$

The next type of load consists of only distributed flexural load acting on the thin-walled beam, and the stochastic varying load is assumed to be following form:

$$f(\xi, t) = f(\xi)g(t), \tag{48}$$

where $g(t)$ is a stochastic process. Note the load considered here is stochastic with respect to time only. The extension to cover spatially varying stochastic load is easy.

The cross-correlation function for the above flexural load is given by

$$R_f(\xi_1, \xi_2, t) = E[f(\xi_1)g(t)f(\xi_2)g(t+\tau)] = f(\xi_1)f(\xi_2)R_g(\tau). \tag{49}$$

The corresponding cross-spectral density function is

$$S_f(\xi_1, \xi_2, \Omega) = f(\xi_1)f(\xi_2)S_g(\Omega). \tag{50}$$

In this case $\eta_{nl}(\Omega)$ simplifies to

$$\eta_{nl}(\Omega) = \int_0^1 f(\xi_1)V_n(\xi_1)\,\mathrm{d}\xi_1 \int_0^1 f(\xi_2)V_l(\xi_2)\,\mathrm{d}\xi_2 S_g(\Omega) = f_n f_l S_g(\Omega). \tag{51}$$

The spectral density functions of the flexural translation, torsional rotation and flexural rotation are then expressed as

$$S_v(\xi, \Omega) = \left| \sum_{l=1}^{\infty} h_l(\Omega) f_l V_l(\xi) \right|^2 S_g(\Omega), \tag{52}$$

$$S_\psi(\xi, \Omega) = \left| \sum_{l=1}^{\infty} h_l(\Omega) f_l \Psi_l(\xi) \right|^2 S_g(\Omega), \tag{53}$$

$$S_\theta(\xi, \Omega) = \left| \sum_{l=1}^{\infty} h_l(\Omega) f_l \Theta_l(\xi) \right|^2 S_g(\Omega). \tag{54}$$

4. Numerical results and discussions

Some numerical results are given to demonstrate the theoretical formulations derived in last section, which can be directly applied to compute the stochastic response of the axially loaded Timoshenko thin-walled beam.

The first example is a cantilever thin-walled beam with monosymmetrical semi-circular cross-section. The geometrical and physical properties of the thin-walled beam shown in Fig. 2 are given as follows: $I = 9.26 \times 10^{-8}\,\mathrm{m}^4$, $J = 1.64 \times 10^{-9}\,\mathrm{m}^4$, $I_s = 0.000501\,\mathrm{kg\,m}$, $y_c = 0.0155\,\mathrm{m}$, $L = 0.82\,\mathrm{m}$, $\Gamma = 1.52 \times 10^{-12}\,\mathrm{m}^6$, $\mu = 0.835\,\mathrm{kg\,m^{-1}}$, $E = 68.9 \times 10^9\,\mathrm{N\,m^{-2}}$, $G = 26.5 \times 10^9\,\mathrm{N\,m^{-2}}$, $k = 0.5$, $A = 3.08 \times 10^{-4}\,\mathrm{m}^2$, $\rho = 2711.04\,\mathrm{kg\,m^{-3}}$, $P = 1790\,\mathrm{N}$.

The natural frequencies and mode shapes of the above axially loaded thin-walled beam for undamped free vibration are computed by setting the damping coefficients c_1, c_2, c_3 and the external excitations $f(x, t)$ and $m(x, t)$ in Eqs. (1)–(3) to zero. The first five natural frequencies of the axially loaded semi-circular section thin-walled beam are shown in Table 1. The corresponding first five normal mode shapes are shown in Figs. 3–5. The first five normal mode shapes with the warping stiffness ignored are shown in Figs. 3(a)–(e). The first five normal mode shapes with the Timoshenko effect omitted are shown in Figs. 4(a)–(e). The first five normal mode shapes including the warping stiffness and Timoshenko effect are shown in Figs. 5(a)–(e). All the first five

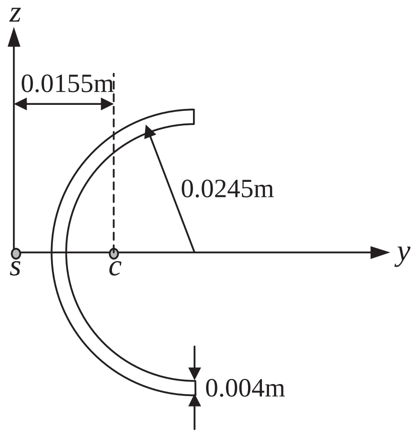

Fig. 2. Beam cross-section used in numerical example 1.

Table 1
Natural frequencies of the cantilever semi-circular section beam with $P = 1790\,\mathrm{N}$

Frequency order	Natural frequency (Hz)			
	Only warping ignored	Only axial force ignored	Only Timoshenko effect ignored	No factors ignored
1	59.97	63.50	61.31	61.03
2	128.12	137.38	136.15	135.86
3	256.00	275.81	275.03	272.47
4	413.07	481.09	479.40	475.72
5	598.55	639.75	661.37	637.36

normal modes are coupled modes, i.e., the modes show coupling between flexural displacement and torsional displacement, although they are dominant torsional modes. The first four mode shapes ignoring the warping stiffness or Timoshenko effect are similar to the ones accounting for the warping stiffness and Timoshenko effect, although there is some difference between them. The fifth mode ignoring the Timoshenko effect is similar to the one accounting for the warping stiffness and Timoshenko effect.

Based on the natural frequencies and mode shapes, the mean square values of the flexural translation, flexural rotation and torsional rotation due to a stochastic varying concentrated flexural load can be computed without any difficulty. The stochastic flexural load is assumed to be an ideal white noise, so the $S_f(\omega)$ in Eqs. (45)–(47) can be replaced by a constant, i.e., $S_f(\omega) = S_0$ (S_0 is a constant). In Figs. 6–8, respectively, are shown the mean square parameters of flexural translation, flexural rotation and torsional rotation along the length of the cantilever thin-walled beam subjected to an ideal white noise concentrated flexural load acting at the tip of the beam. The value of the damping coefficient used in computation is 0.01. The mean square flexural translation accounting for the Timoshenko effect is only a little different from the one excluding the Timoshenko effect. But the effect of Timoshenko on the flexural rotation and torsional

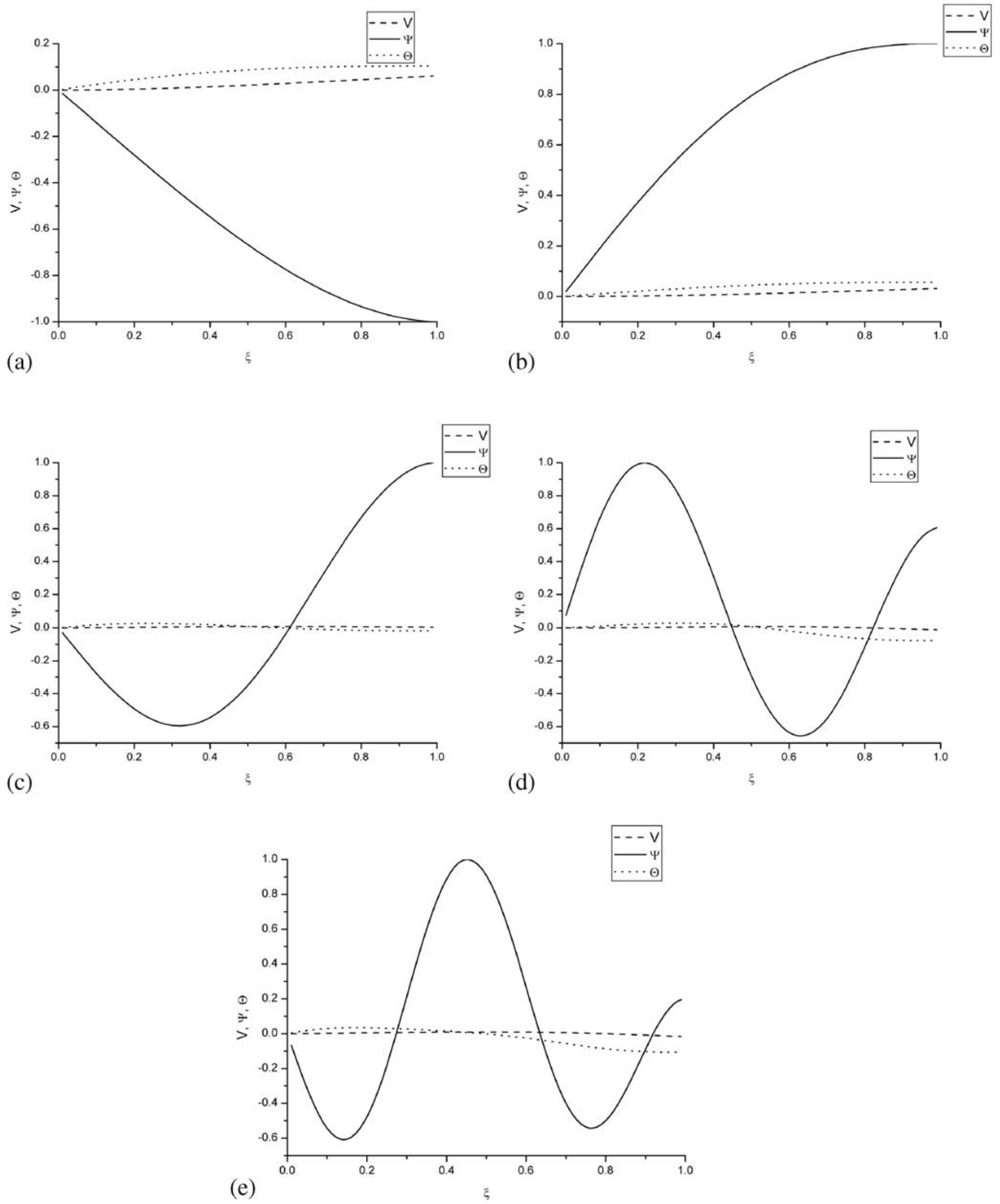

Fig. 3. First five normal mode shapes of example 1 excluding the effect of warping stiffness: (a) mode 1; (b) mode 2; (c) mode 3; (d) mode 4; (e) mode 5.

rotation is noticeable. The mean square flexural and torsional displacements including the warping stiffness are remarkably different from the ones excluding the warping stiffness, as can be seen from Figs. 6–8. Compared to the effects of Timoshenko and warping stiffness, the axial force has a more significantly effect on the mean square flexural and torsional displacements. The percentage errors for mean square values of the flexural and torsional response at the tip of the

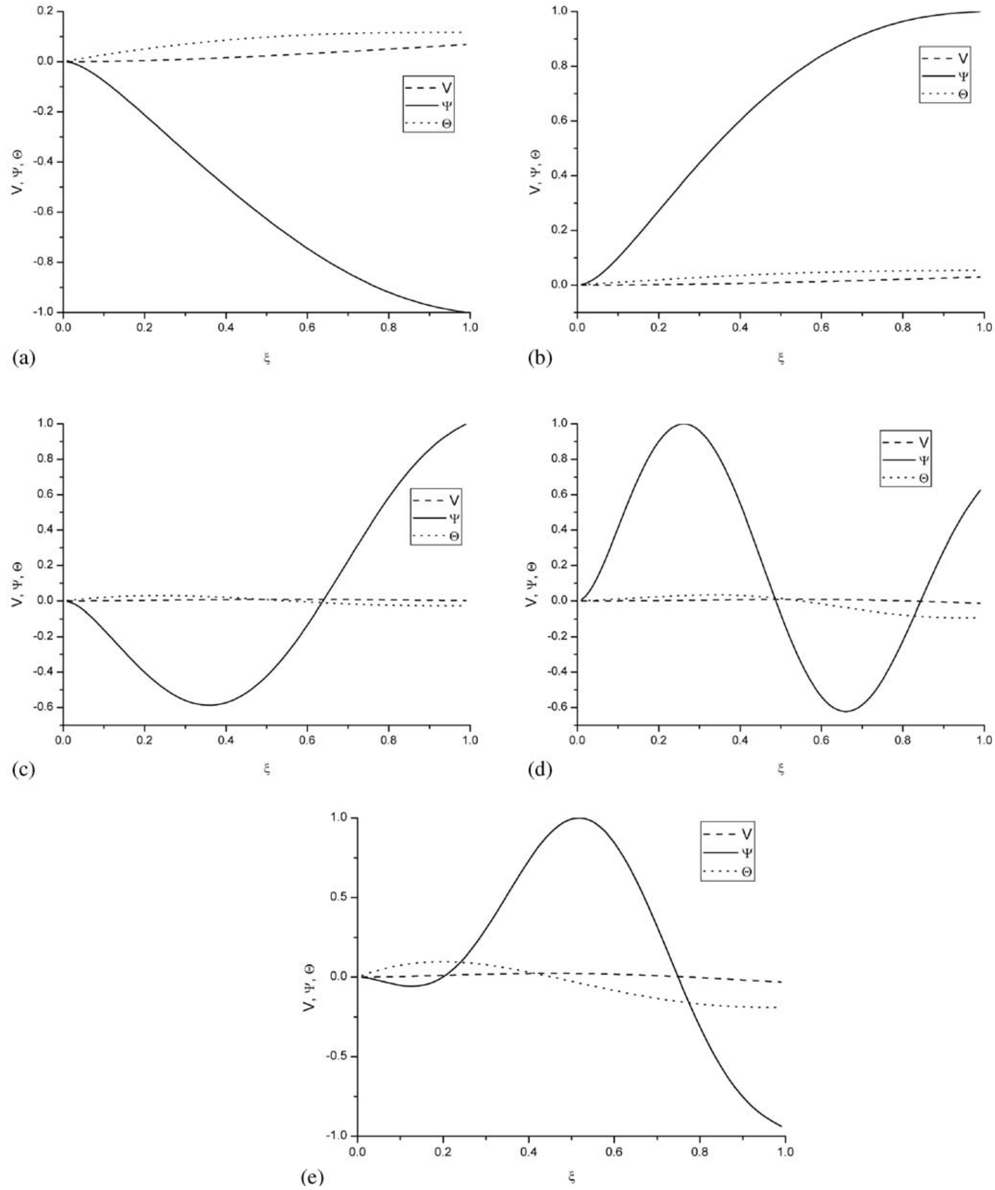

Fig. 4. First five normal mode shapes of example 1 excluding the Timoshenko effect: (a) mode 1; (b) mode 2; (c) mode 3; (d) mode 4; (e) mode 5.

cantilever thin-walled beam without the warping stiffness or Timoshenko effect or axial force included are shown in Table 2. The numerical results show that it is necessary to consider the effects of the axial force, warping stiffness and Timoshenko in order to obtain the mean square displacement response accurately.

A cantilever thin-walled uniform beam with a monosymmetrical channel cross-section is considered next. The geometrical properties and physical properties of the beam shown in Fig. 9

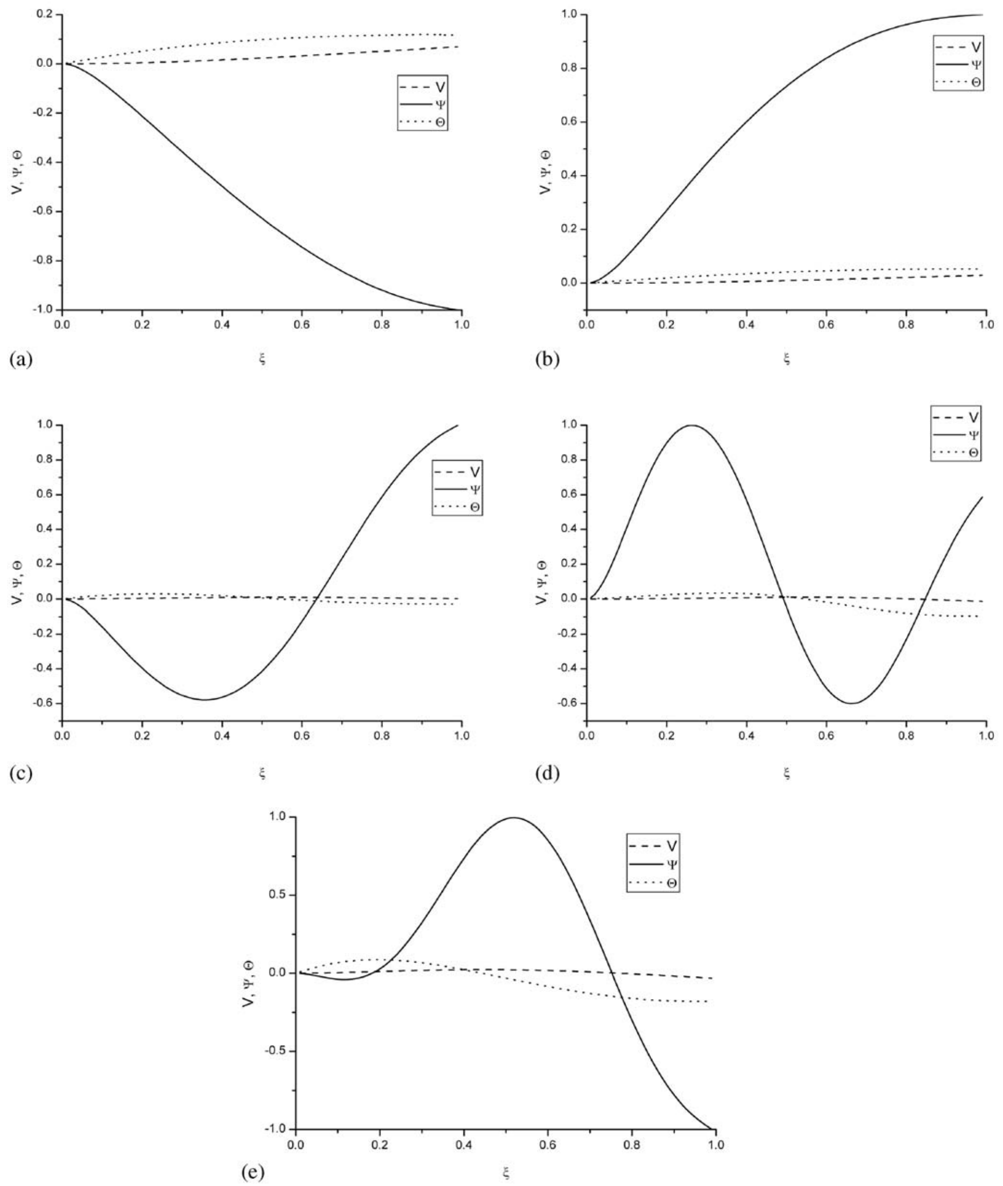

Fig. 5. First five normal mode shapes of example 1 including the warping stiffness and Timoshenko effect: (a) mode 1; (b) mode 2; (c) mode 3; (d) mode 4; (e) mode 5.

are given below: $I = 1.449 \times 10^{-3}\,\mathrm{m}^4$, $J = 1.223 \times 10^{-5}\,\mathrm{m}^4$, $I_s = 56.87\,\mathrm{kg\,m}$, $y_c = 0.336\,\mathrm{m}$, $L = 3.2\,\mathrm{m}$, $\Gamma = 3.885 \times 10^{-5}\,\mathrm{m}^6$, $\mu = 225\,\mathrm{kg\,m}^{-1}$, $E = 2.1 \times 10^{11}\,\mathrm{N\,m}^{-2}$, $G = 8 \times 10^{10}\,\mathrm{N\,m}^{-2}$, $k = 0.5136$, $A = 0.012856\,\mathrm{m}^2$, $\rho = 17501.6\,\mathrm{kg\,m}^{-3}$, $P = 1.85 \times 10^6\,\mathrm{N}$.

The first five natural frequencies of the axially loaded channel section thin-walled beam are shown in Table 3. The corresponding mode shapes of the first five normal modes with an axial

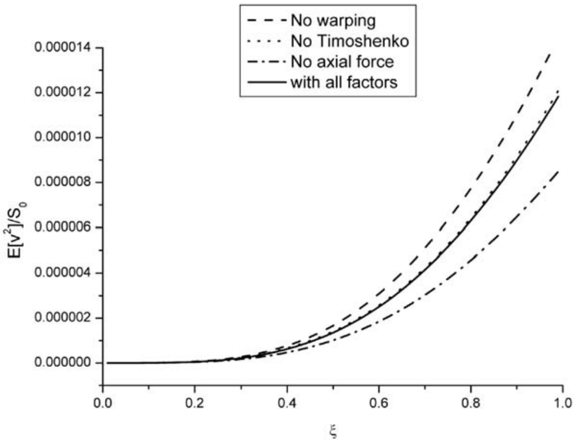

Fig. 6. Mean square flexural translation along the length of the cantilever thin-walled beam.

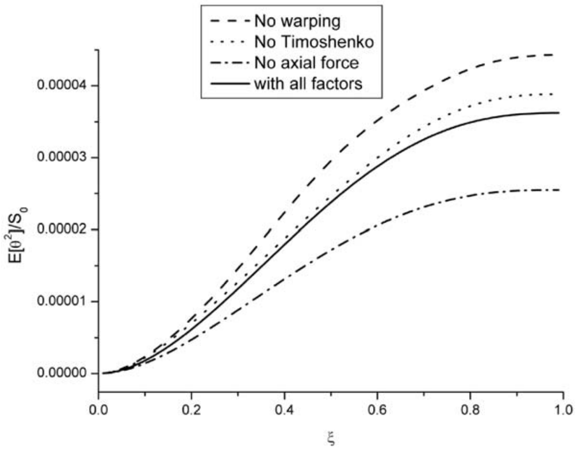

Fig. 7. Mean square flexural rotation along the length of the cantilever thin-walled beam.

force $P = 1.85 \times 10^6$ N are plotted in Figs. 10–12. The first five normal mode shapes disregarding the warping stiffness effect are plotted in Figs. 10(a)–(e). The first five normal mode shapes with the Timoshenko effect ignored are plotted in Figs. 11(a)–(e). The first five normal mode shapes taking into account the effects of warping stiffness and Timoshenko are plotted in Figs. 12(a)–(e). It can be seen from Fig. 10 that the first five normal modes omitting the effect of warping stiffness are absolutely dominant torsional modes. The warping stiffness and Timoshenko effect make great difference to all of the first five mode shapes. It can be seen from Figs. 11 and 12 that all of the first five modes are coupled modes. The mode 1, mode 3, mode 4 are dominant torsional modes when the Timoshenko effect is excluded, and the mode 2 and mode 5 excluding the Timoshenko effect are strongly coupled modes. When the warping stiffness and Timoshenko effect are considered, the mode 1, mode 2 and mode 3 are dominant torsional modes, but the mode 4 and mode 5 are strongly coupled modes.

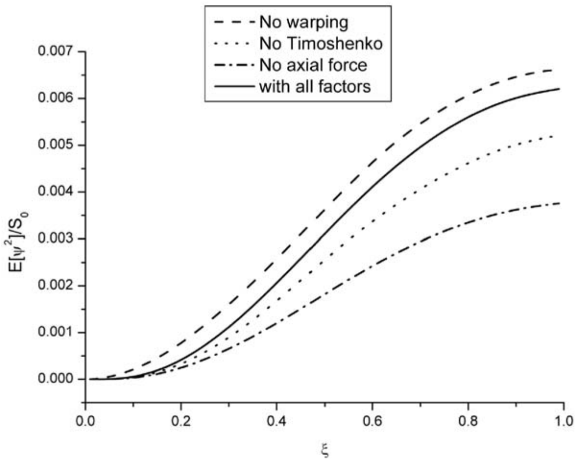

Fig. 8. Mean square torsional rotation along the length of the cantilever thin-walled beam.

Table 2
Percentage errors for mean square values of the flexural and torsional response at the tip of the cantilever thin-walled beam

	Warping ignored (%)	Timoshenko ignored (%)	Axial force ignored (%)
Flexural translation	22.88	2.54	−28.05
Flexural rotation	22.38	7.18	−29.56
Torsional rotation	6.61	−15.97	−39.52

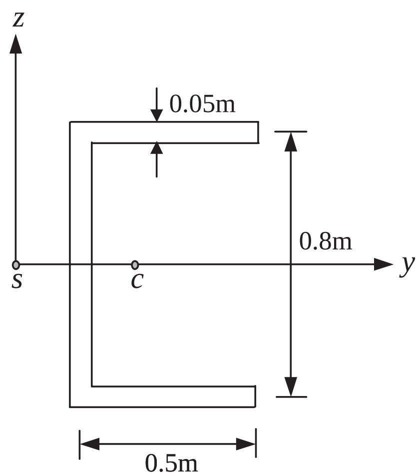

Fig. 9. Beam cross-section used in numerical example 2.

Table 3
Natural frequencies of the cantilever channel section beam with $P = 1.85 \times 10^6$ N

Frequency order	Natural frequency (Hz)			
	Only warping ignored	Only axial force ignored	Only Timoshenko effect ignored	No factors ignored
1	7.30	23.78	22.05	21.82
2	21.91	77.24	87.99	76.74
3	36.81	124.77	128.88	122.25
4	51.55	295.25	356.37	293.84
5	66.32	334.87	549.21	333.32

According to the procedure discussed before, the stochastic response of the above axially loaded thin-walled beam can be computed without any difficulty based on the natural frequencies and mode shapes. To compare the results obtained from the present theory including the effects of warping stiffness, Timoshenko and axial force with those given by the theory excluding the effect of warping stiffness or Timoshenko effect or axial force effect, the mean square values of the flexural translation, flexural rotation and torsional rotation due to a stochastic varying concentrated flexural load are calculated. In Figs. 13–15, respectively, are shown the mean square parameters of the flexural translation, flexural rotation and torsional rotation along the length of the channel section thin-walled beam subjected to an ideal white noise concentrated flexural load acting at the tip of the beam. The value of the damping coefficient has been taken as 0.01. It can be seen from Figs. 13–15, the mean square values of the flexural displacement and torsional displacement predicted by the present theory considering the effects of axial force, Timoshenko and warping stiffness are distinctly different from those obtained from the theory excluding the effect of axial force or warping stiffness or Timoshenko effect. So, it is absolutely necessary to include the effects of axial force, warping stiffness and Timoshenko effect when the mean square displacements of this particular thin-walled beam are computed. The percentage errors for mean square values of the flexural and torsional response at the tip of the cantilever thin-walled beam without the warping stiffness or Timoshenko effect or axial force included are shown in Table 4.

5. Conclusions

A method has been presented to perform the stochastic response analysis of the axially loaded Timoshenko thin-walled beam. The thin-walled beam is assumed to be uniform, straight, damped, and subjected to an axial force. The effects of shear deformation, rotatory inertia and warping stiffness, which are usually important for open cross-section thin-walled beam, are included in the present formulations. Once the natural frequencies and mode shapes of the axially loaded thin-walled beam are obtained, mode superposition method in conjunction with receptance method is used to compute the stochastic response of the beam such as the flexural displacement and torsional displacement. Although the illustrative examples given in this paper are that of two

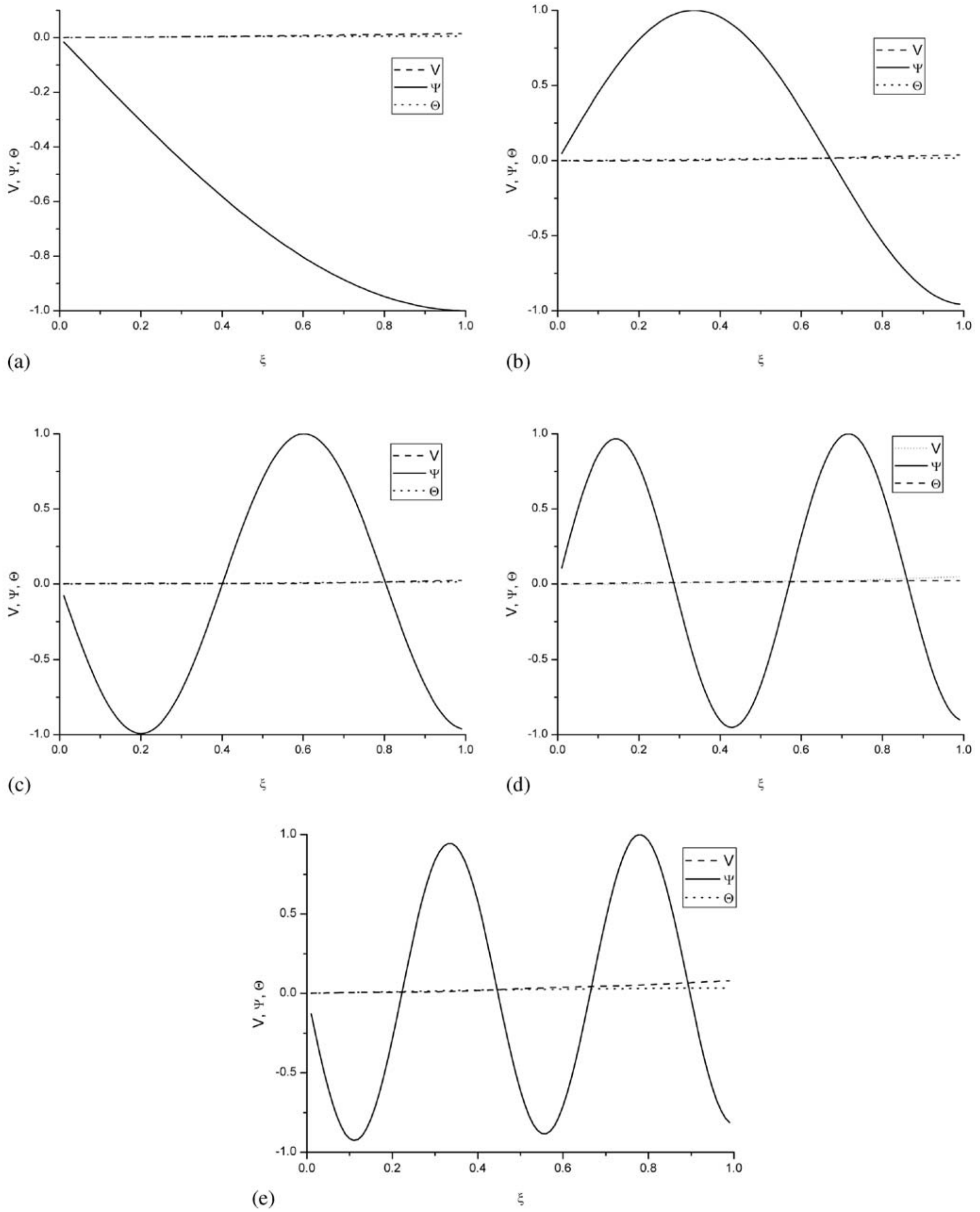

Fig. 10. First five normal mode shapes of example 2 without the warping stiffness included: (a) mode 1; (b) mode 2; (c) mode 3; (d) mode 4; (e) mode 5.

simple thin-walled beams, the developed theory can be applied to other types of boundary conditions of the thin-walled beams or beam assemblages and can be used to other kinds of stochastic excitations.

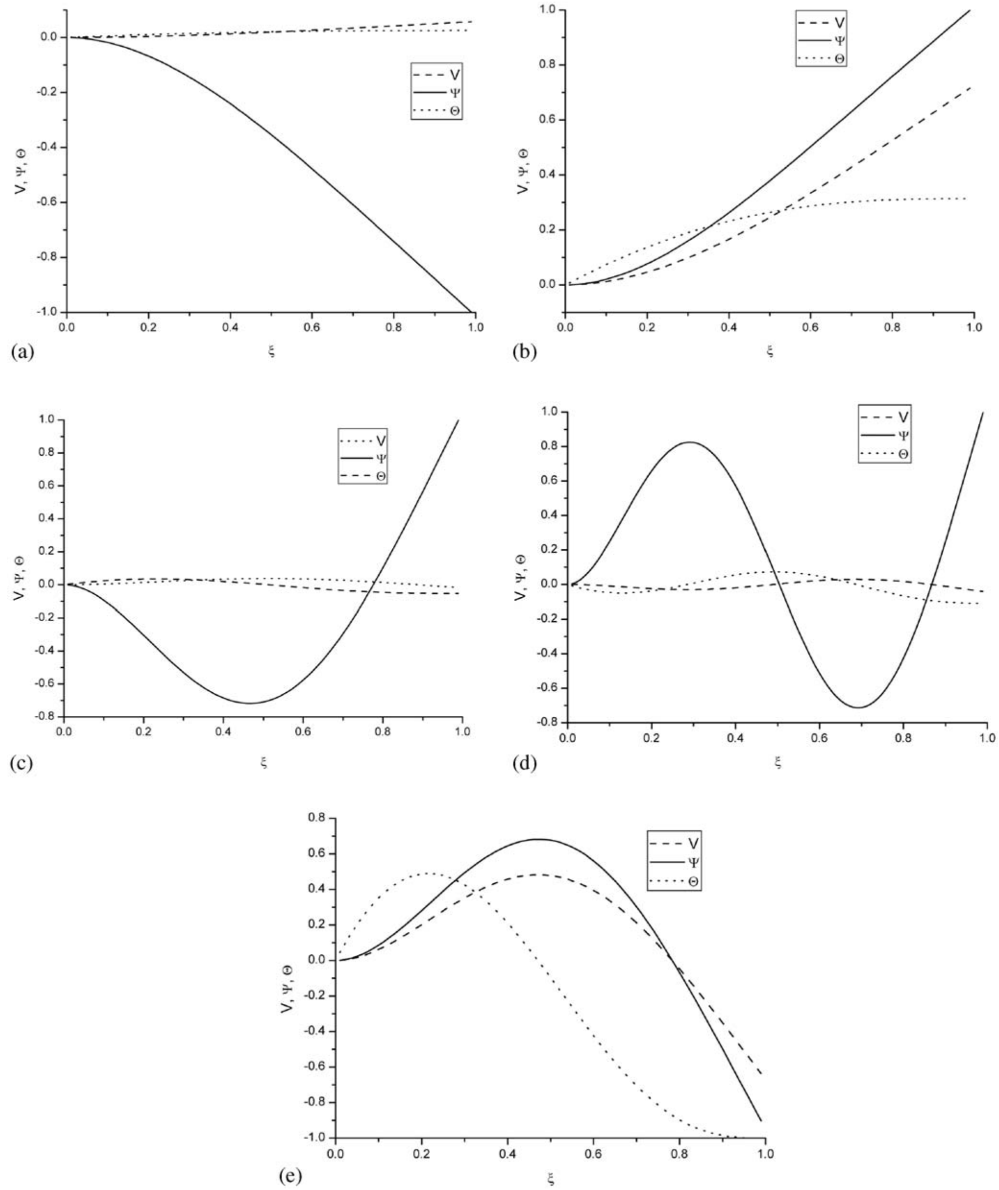

Fig. 11. The first five normal mode shapes of example 2 excluding the Timoshenko effect: (a) mode 1; (b) mode 2; (c) mode 3; (d) mode 4; (e) mode 5.

Appendix A

The damped governing equations of motion for the forced vibration of the axially loaded Timoshenko thin-walled beam exhibiting flexure–torsion coupling and including warping stiffness effect can be derived using the Hamilton's principle as follows.

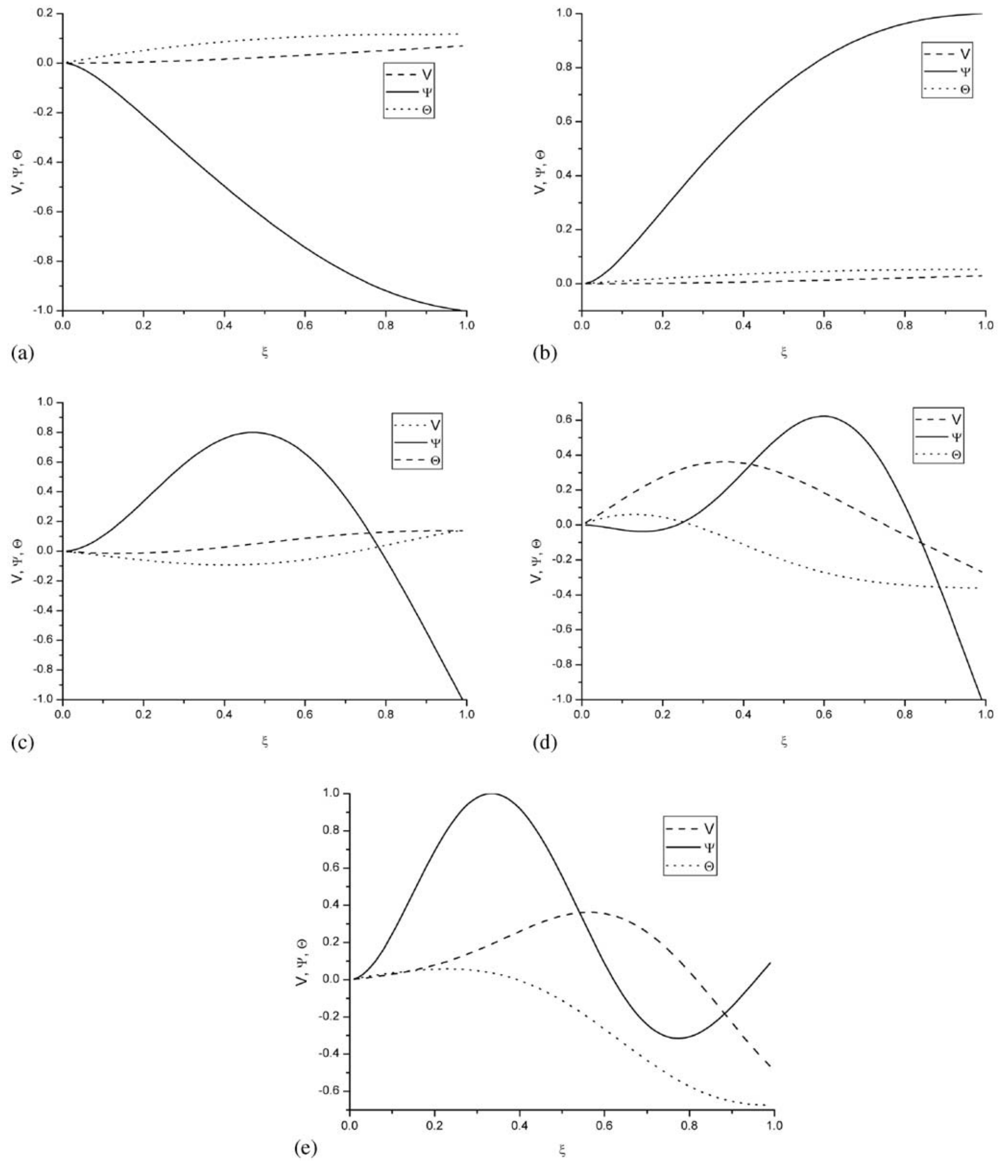

Fig. 12. The first five normal mode shapes of example 2 including the warping stiffness and Timoshenko effect: (a) mode 1; (b) mode 2; (c) mode 3; (d) mode 4; (e) mode 5.

The total strain energy U of an axially loaded Timoshenko thin-walled beam shown in Fig. 1 is given by

$$U = \frac{1}{2}\int_0^L \{EI(\theta')^2 - P[(v')^2 - 2y_c v'\psi' + (I_s/\mu)(\psi')^2] + kAG(v'-\theta)^2 + E\Gamma(\psi'')^2 + GJ(\psi')^2\}\,\mathrm{d}x, \tag{A.1}$$

where all the variables and symbols are defined in Section 2.

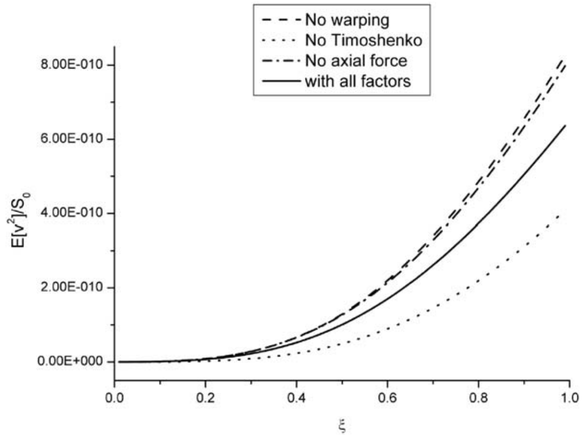

Fig. 13. Mean square flexural translation along the length of the cantilever thin-walled beam.

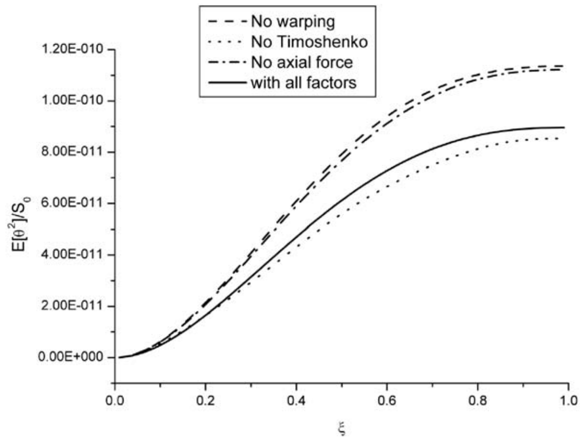

Fig. 14. Mean square flexural rotation along the length of the cantilever thin-walled beam.

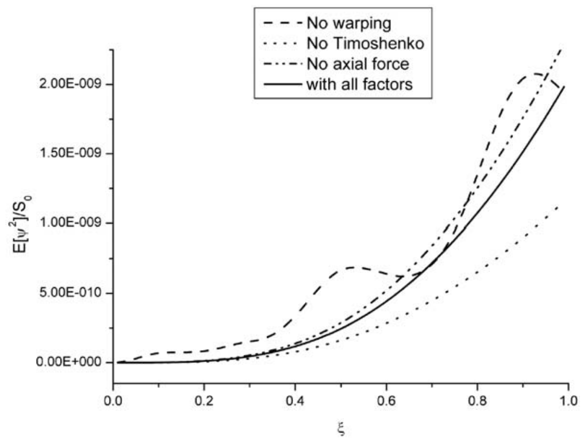

Fig. 15. Mean square torsional rotation along the length of the cantilever thin-walled beam.

Table 4
Percentage errors for mean square values of the flexural and torsional response at the tip of the cantilever thin-walled beam

	Warping ignored (%)	Timoshenko ignored (%)	Axial force ignored (%)
Flexural translation	29.67	−35.79	25.27
Flexural rotation	27.23	−4.80	25.00
Torsional rotation	−2.02	−41.41	16.16

The total kinetic energy T of an axially loaded Timoshenko thin-walled beam is given by

$$T = \frac{1}{2}\int_0^L [\mu(\dot{v}^2 - 2y_c\dot{v}\dot{\psi}) + I_s\dot{\psi}^2 + \rho I\dot{\theta}^2]\,\mathrm{d}x. \tag{A.2}$$

The governing equations of motion and the boundary conditions can be derived conveniently by means of Hamilton's principle, which can be stated in the form

$$\int_{t_1}^{t_2} (\delta T - \delta U + \delta W)\,\mathrm{d}t = 0, \tag{A.3}$$

$\delta v = \delta\theta = \delta\psi = \delta\psi' = 0$ at $t = t_1, t_2$.

Herein T is the kinetic energy, U the potential energy, δW the virtual work of the non-conservative forces, which can be written as

$$\delta W = \int_0^L [f(x,t)\delta v + m(x,t)\delta\psi - c_1(\dot{v} - y_c\dot{\psi})\delta v - (c_2\dot{\psi} - c_1 y_c\dot{v})\delta\psi - c_3\dot{\theta}\delta\theta]\,\mathrm{d}x. \tag{A.4}$$

Substituting Eqs. (A.1), (A.2) and (A.4) into Eq. (A.3) and carrying out the usual steps yield the governing equations of motion and the boundary conditions.

(a) The governing equations of motion

$$\mu\ddot{v} - \mu y_c\ddot{\psi} + Pv'' - Py_c\psi'' - kAGv'' + kAG\theta' + c_1(\dot{v} - y_c\dot{\psi}) = f(x,t), \tag{A.5}$$

$$I_s\ddot{\psi} - \mu y_c\ddot{v} - Py_cv'' + P(I_s/\mu)\psi'' + E\Gamma\psi'''' - GJ\psi'' + (c_2\dot{\psi} - c_1y_c\dot{v}) = m(x,t), \tag{A.6}$$

$$\rho I\ddot{\theta} - EI\theta'' - kAGv' + kAG\theta + c_3\dot{\theta} = 0. \tag{A.7}$$

(b) The boundary conditions at the ends ($x = 0, L$)

$$(Pv' - Py_c\psi' - kAGv' + kAG\theta)\delta v = 0, \tag{A.8}$$

$$(-Py_cv' + P\psi' I_s/\mu + E\Gamma\psi''' - GJ\psi')\delta\psi = 0, \tag{A.9}$$

$$(-EI\theta')\delta\theta = 0, \tag{A.10}$$

$$(-E\Gamma\psi'')\delta\psi' = 0. \tag{A.11}$$

References

[1] F.Y. Cheng, C.P. Pantelides, Dynamic Timoshenko beam-columns on elastic media, *American Society of Civil Engineers, Journal of Structural Engineering* 114 (1988) 1524–1550.

[2] R. Nagem, Random vibration of point driven beams, *Journal of Sound and Vibration* 151 (1991) 168–174.

[3] M.P. Singh, A.S. Abdelnaser, Random vibration of externally damped viscoelastic Timoshenko beams with general boundary conditions, *Transactions of the American Society of Mechanical Engineers, Journal of Applied Mechanics* 60 (1993) 149–156.

[4] T.P. Chang, Deterministic and random vibration of an axially loaded Timoshenko beam resting on elastic foundation, *Journal of Sound and Vibration* 178 (1994) 55–66.

[5] R.E.D. Bishop, W.G. Price, Coupled bending, twisting of a Timoshenko beam, *Journal of Sound and Vibration* 50 (1977) 469–477.

[6] W.L. Hallauer Jr., R.Y.L. Liu, Beam bending–torsion dynamic stiffness method for calculation of exact vibration modes, *Journal of Sound and Vibration* 85 (1982) 105–113.

[7] P.O. Friberg, Coupled vibration of beams – An exact dynamic element stiffness matrix, *International Journal for Numerical Methods in Engineering* 19 (1983) 479–493.

[8] P.O. Friberg, Beam element matrices derived from Vlasov's theory of open thin-walled elastic beams, *International Journal for Numerical Methods in Engineering* 21 (1985) 1205–1228.

[9] E. Dokumaci, An exact solution for coupled flexural and torsional vibrations of uniform beams having single cross-sectional symmetry, *Journal of Sound and Vibration* 119 (1987) 443–449.

[10] R.E.D. Bishop, S.M. Cannon, S. Miao, On coupled bending and torsional vibration of uniform beams, *Journal of Sound and Vibration* 131 (1989) 457–464.

[11] E.N. Dvorkin, D. Celentano, A. Cuitino, G. Gioia, A Vlasov beam element, *Computers and Structures* 33 (1989) 187–196.

[12] A.Y.T. Leung, Natural shape functions of a compressed Vlasov element, *Thin-walled Structures* 11 (1991) 431–438.

[13] A.Y.T. Leung, Dynamic stiffness analysis of thin-walled structures, *Thin-walled Structures* 14 (1992) 209–222.

[14] J.R. Banerjee, F.W. Williams, Coupled bending–torsional dynamic stiffness matrix for Timoshenko beam elements, *Computers and Structures* 42 (1992) 301–310.

[15] J.R. Banerjee, F.W. Williams, Coupled bending–torsional dynamic stiffness matrix of an axially loaded Timoshenko beam element, *International Journal of Solids and Structures* 31 (1994) 749–762.

[16] M.J. Klausbruckner, R.J. Pryputniewicz, Theoretical and experimental study of coupled vibrations of channel beams, *Journal of Sound and Vibration* 183 (1995) 239–252.

[17] J.R. Banerjee, S. Guo, W.P. Howson, Exact dynamic stiffness matrix of a bending–torsion coupled beam including warping, *Computers and Structures* 59 (1996) 613–621.

[18] A.N. Bercin, M. Tanaka, Coupled flexural–torsional vibrations of Timoshenko beams, *Journal of Sound and Vibration* 207 (1997) 47–59.

[19] M. Tanaka, A.N. Bercin, Free vibration solution for uniform beams of nonsymmetrical cross section using Mathematica, *Computers and Structures* 71 (1999) 1–8.

[20] S. Mohammad Hashemi, Marc J. Richard, Free vibrational analysis of axially loaded bending–torsion coupled beams: a dynamic finite element, *Computers and Structures* 77 (2000) 711–724.

[21] X. Chen, K.K. Tamma, Dynamic response of elastic thin-walled structures influenced by coupling effects, *Computers and Structures* 51 (1994) 91–105.

[22] S.H.R. Eslimy-Isfahany, J.R. Banerjee, A.J. Sobey, Response of a bending–torsion coupled beam to deterministic and random loads, *Journal of Sound and Vibration* 195 (1996) 267–283.